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REPORT

THE EFFECT OF YAW UPON AIRCRAFT GUNFIRE TRAJECTORIES

BY

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THE EFFECT OF YAW UPON AIRCRAFT GUNFIRE TRAJECTORIES

Abstract

Based on the analysis by Fowler, Gallop, Lock and Richmond of the motion of a spinning projectile, equations are derived for the computation of the trajectories, influenced by yaw, of bullets fired from moving aircraft. The equations are applicable to bullets whose precessional and nutational damping rates are equal, and thus to all small arms bullets whose damping rates have been experimentally determined, as well as to 37mm shell. An algebraic formula is derived for the amount of windage jump, and procedures are described for the computation of aircraft firing tables.

1. Introduction. The motion of a bullet about its center of mass influences the motion of its center of mass. Since the initial conditions of projection of bullets fired from a moving aircraft are in general such as to produce motion about the center of mass, it is therefore necessary to consider the motion about the center of mass in the computation of aircraft firing tables. The author's investigations having led to formulae useful in the computation of such tables, his results are presented here. The motion about the center of mass influences the firing tables (1) through an increased drag and (2) through the "windage jump".

2. Trajectories When Yaw Effects Are Neglected. The equations for computing aircraft gunfire trajectories when yaw effects are neglected have been brought into a most advantageous form by L. E. Cunningham. His equations may be derived as follows. Let x, y, z be a right-handed set of orthogonal axes moving with the aircraft and having their origin at the muzzle of the gun, the x axis pointing vertically upward, the z axis pointing in the direction of flight of the aircraft (taken to be horizontal), and the y axis pointing horizontally to starboard. Let the azimuth of the bore of the gun be denoted by A, measured in the plane yz from z through y, and denote the zenith distance of the bore of the gun by Z. Then the direction cosines of the bore are $\cos Z$, $\sin Z \sin A$, and $\sin Z \cos A$. Let x', y', z' be a set of axes parallel to the first set, at rest in the air, and such that the origins coincide at the instant of

firing. The initial components of velocity of the bullet, in the primed system, are thus $\underline{v}_o \cos \underline{Z}$, $\underline{v}_o \sin \underline{Z} \sin \underline{A}$, and $\underline{v}_o \sin \underline{Z} \cos \underline{A} + \underline{w}$, where $\underline{v}_o$ is the muzzle velocity and $\underline{w}$ is the true air speed of the aircraft. It follows that the initial true air speed of the bullet is u_o, given by

$$u_o^2 = v_o^2 + 2wv_o \sin Z \cos A + w^2 \tag{1}$$

and that if after a time of flight $\underline{t}$ the Siacci coordinates of the bullet in the primed system of axes are P, Q, then the coordinates of the bullet in the system x,y,z moving with the aircraft are

$$(Pv_o/u_o) \cos Z - Q,\ (Pv_o/u_o) \sin Z \sin A,\ (P/u_o)(v_o \sin Z \cos A + w) - wt.$$

Here P is the distance, measured along the line of departure in the primed system of axes, from the origin to a point directly above the bullet; the "drop" Q is the distance from the point to the bullet.

Let ξ, η, ζ be orthogonal axes moving with the aircraft; ξ, along the bore axis, η in a vertical plane through the bore axis and directed away from the ground, and ζ horizontal and pointing to the right of the bore axis. The relation between the x,y,z and ξ, η, ζ axes is given by the scheme,

	x	y	z
ξ	cos Z	sin Z sin A	sin Z cos A
η	sin Z	-cos Z sin A	-cos Z cos A
ζ	0	cos A	-sin A

and thus

$$\begin{aligned} \xi &= -\tau \sin Z \cos A - Q \cos Z + Pv_o/u_o \\ \eta &= \tau \cos Z \cos A - Q \sin Z \\ \zeta &= \tau \sin A \end{aligned} \tag{2}$$

where

$$\tau = w(t - \frac{P}{u_o}).$$

The numerical procedure in computing an undisturbed trajectory is merely to compute $\underline{t}$, and $\underline{Q}$ for any given $\underline{P}$. Then τ is computed, and the equations (2) give ξ, η, and ζ. The slant range, $\underline{D}$, is obtained from

$$D^2 = \xi^2 + \eta^2 + \zeta^2 \tag{3}$$

and then the lateral deflection, λ, and the vertical deflection, μ, in angular measure, are obtained from their definitions thus

$$\begin{aligned} \sin \lambda &= \zeta/D \\ \sin \mu &= \eta/D. \end{aligned} \tag{4}$$

The quantity λ is defined as the angle between the line gun-bullet and the vertical plane through the bore, and is positive when the line is to the right of the plane; the quantity μ is defined as the angle between the same line and the "slant" plane through the bore and perpendicular to the vertical plane, and μ is positive when the line lies above the slant plane. The computations are carried out for several values of P, and then by interpolation furnish values of λ, μ, and $\underline{t}$ as functions of $\underline{D}$.

It will be noticed that $\underline{t}$ and $\underline{Q}$ must be computed for a known value of $\underline{P}$. The most rapid procedure is to use the tabulated Siacci functions S, T, I, and A of the pseudo-velocity $u = \bar{P}$, thus:

$$S = S_o + (\rho/c)P \tag{5}$$

$$t = (c/\rho)(T - T_o) \tag{6}$$

$$Q = (c^2/2\rho^2)(A - A_o) - (c/2\rho)I_oP \tag{7}$$

where the subscripts "$_o$" refer to the argument u_o. Equation (5) furnishes $\underline{u}$, and then (6) and (7) yield $\underline{t}$ and $\underline{Q}$, respectively. In these equations $\underline{c}$ is the appropriate ballistic coefficient, and ρ is the relative air density, referred to a standard of 0.07513 pounds per cubic foot. For aircraft firing tables it has been regarded as sufficiently accurate to ignore the variation of ρ along a trajectory, and to use the value that exists at the gun.

3. The Motion of a Bullet about its Center of Mass. The general discussion of Section 3 of this Report is independent of Section 2, and in order to conform to the Ballistic Research Laboratory's older notations, it will be necessary to use some of the same symbols as in Section 2 for new quantities. Consider a set of right-handed axes moving with the tangent to the mean* trajectory, so that the axis $\underline{01}$ is the tangent to the mean trajectory drawn in the direction of motion, while $\underline{02}$ is the upward normal and $\underline{03}$ is horizontal and to the right, as viewed from the gun. Denote the direction cosines of the bullet's axis by l, m^*, n. The motion about the center of mass consists (1) of a secular part related to "drift", ignorable in aircraft firing tables, and (2) of a vibratory motion. The vibratory motion about the center of mass is associated with a vibration of the center of mass, with respect to the mean trajectory; denote the direction cosines of the velocity vector of the center of mass by x, y, z. One may set $m^* - y = j$, $n - z = k$. For small angles of yaw, one has very nearly

$$\begin{aligned} j &= \delta \cos\varphi \\ k &= \delta \sin\varphi \end{aligned} \tag{8}$$

* The "mean trajectory" is a particle trajectory that differs from the actual trajectory only by periodic terms, damped or undamped.

where δ is the angle of yaw, and φ is the angle of orientation of the yaw. The angles δ and φ are referred to the instantaneous direction of motion of the center of mass, and are appropriate to yaw-card measurements. The angles δ' and φ' measured by spark photography, on the other hand, are referred to the mean trajectory and those angles are related to the m* and n in the same way as δ and φ are related to j and k. As will later be seen, the difference between the two ways of measuring the yaws involves only y and z, which are of the order for most projectiles of only a few percent of the yaw angles. The quantities j and k may be called the "rectangular components" of the yaw.

It follows from the results of Fowler** (their equation 4.01) et al, that the motion is given by a linear combination of the solutions

$$\begin{aligned} j &= \cos n_1 t, & k &= \sin n_1 t; \\ j &= \sin n_1 t, & k &= -\cos n_1 t; \\ j &= \cos n_2 t, & k &= \sin n_2 t; \\ j &= \sin n_2 t, & k &= -\cos n_2 t \end{aligned} \tag{9}$$

with four coefficients that are slowly varying functions of the time. Here

$$\begin{aligned} n_1 &= (AN/2B)(1 + p) \\ n_2 &= (AN/2B)(1 - p) \end{aligned} \tag{10}$$

where

$$p = \left(1 - \frac{1}{s}\right)^{\frac{1}{2}}.$$

In equations (10), A and B are respectively the axial and transverse moments of inertia, N is the spin in radians per second reckoned positive if right-handed, and s is the stability factor given by

$$s = A^2N^2/4B\mu$$

where μ is the moment factor

$$\mu = \rho_a u^2 d^3 K_M$$

in which ρ_a is the air density, u is the velocity of the bullet, K_M is the overturning moment coefficient, and d is the diameter of the bullet.

** Fowler, Gallop, Lock and Richmond, Phil. Trans. Royal Soc. A, Vol. 221, p. 295, 1920.

The slow variation of the coefficients of the periodic terms proceeds as follows. The coefficients of the terms involving n_1 consist of constant factors, arbitrarily disposable, multiplied by the damping factor

$$p^{-\frac{1}{2}} e^{-\int_0^t \left(\frac{f + \varkappa}{2} + \frac{f - \varkappa + 2\gamma}{2p}\right) dt}$$

and the coefficients of the terms involving n_2 consist of arbitrarily disposable constant factors, multiplied by the damping factor

$$p^{-\frac{1}{2}} e^{-\int_0^t \left(\frac{f + \varkappa}{2} - \frac{f - \varkappa + 2\gamma}{2p}\right) dt}$$

where

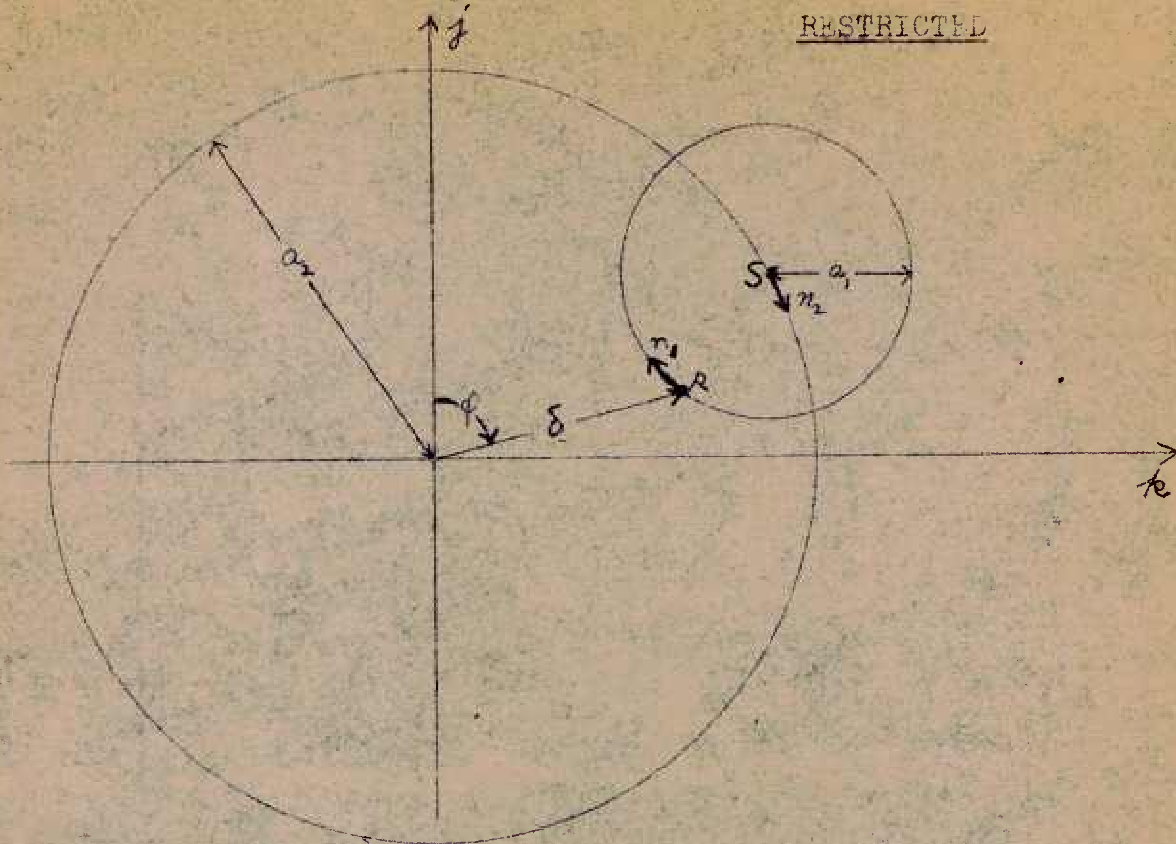
$$f = \rho_a u d^4 K_H / B$$

$$\varkappa = \rho_a u d^2 K_L / m$$

$$\gamma = \rho_a u d^4 K_J / A.$$

In the preceding equations $\underline{m}$ is the mass of the projectile, and K_H, K_L, and K_J are the dimensionless yawing moment, cross wind force, and Magnus moment coefficients.

The motion about the center of mass has a simple geometrical interpretation, which facilitates one's clear conception of the nature of the yawing motion. Ignore for the moment the gradual variation of the coefficients of the terms (9). Then $\underline{j}$, $\underline{k}$ are the rectangular coordinates of a point $\underline{R}$ which is rotating in a clockwise direction at an angular rate n_1 (algebraically) in a circular path of radius a_1, whose center is a point $\underline{S}$ that describes a circle of radius a_2 clockwise around the origin at an angular rate n_2 (algebraically).



The clockwise angle from the j-axis to the radius vector to R is the angle of orientation of the yaw, φ; and the distance of R from the origin is the angle of yaw, δ. It is seen that the motion is merely epicyclic, to the accuracy of Fowler *et al.*'s analysis. The part of the motion involving n_1, namely the rotation in a circle of radius a_1 at a rate n_1, may be called the "nutation"; the rotation in a circle of radius a_2 at the slower rate n_2 may be called "precession"; the resultant of the nutation and precession is the complete yawing motion. The periodic variation of the angle δ of course involves $n_1 - n_2$ thus

$$\delta^2 = a_1^2 + a_2^2 + 2a_1a_2\cos(n_1 - n_2)t \tag{11}$$

if the time is measured from a suitable instant. One may call a_1 the "amplitude" of the nutation, and a_2 the amplitude of the precession. The slow variation of the coefficients, described on the previous page, involves merely a gradual variation of the two amplitudes, at rates which are in general different for the two, and does not otherwise alter the geometrical interpretation that has been given.

The rates of variation, or of "damping", of the amplitudes may be experimentally determined, by yaw-cards or spark photographs. The maximum angle of yaw is clearly $a_1 + a_2$, and the minimum angle of yaw is clearly $|a_1 - a_2|$. Now it has been experimentally determined*, for all the small arms bullets that have been so studied as well as for a 37mm projectile, that if

* By H. P. Hitchcock

the minimum yaw $|a_1 - a_2|$ is initially zero, it remains zero. This can be true only if the precessional and nutational damping rates, of those bullets, are equal; for in the experiments $a_1 + a_2$ was not zero. It follows that for such bullets $f - \varkappa + 2\gamma = 0$, and that both damping factors may be written

$$p^{-\frac{1}{2}} e^{-\rho_a a P}$$

where P is the distance the bullet has travelled, and is very nearly equal to the Siacci "P", for aircraft trajectories. The constant <u>a</u> may be experimentally determined, and is independent of velocity and air density to the extent that K_H and K_L are so independent:

$$a = d^4 K_H/2B + d^2 K_L/2m. \tag{11A}$$

4. <u>Vibrations of the Center of Mass. The "Windage Jump".</u> Equations (9) of the last section, and the discussion of the damping rates, show that the complete solution of the vibratory motion may be written as

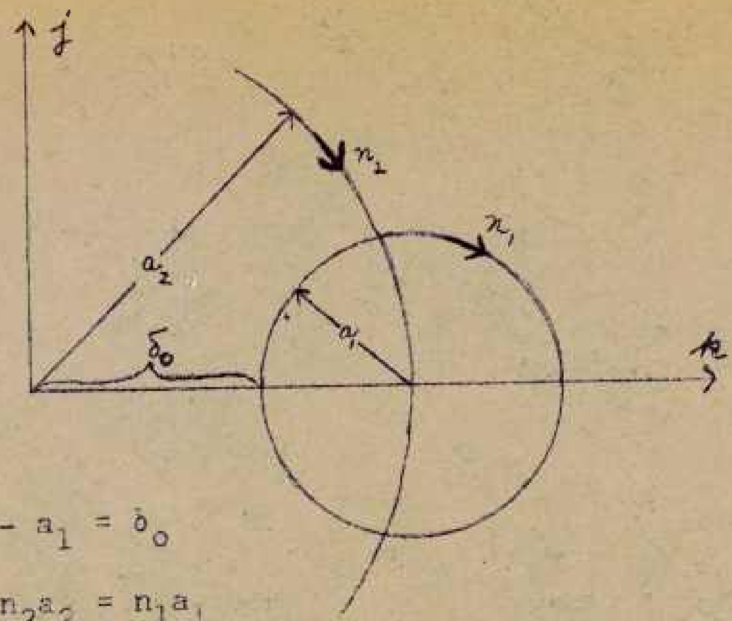
$$\eta = K_1 e^{in_1 t - \lambda_1 t} + K_2 e^{in_2 t - \lambda_2 t}$$

where K_1 and K_2 are two complex constants, the λ's correspond to the damping rates per second, and where the real part of η is <u>j</u> while the complex part is <u>k</u>. Provided, as is true for aircraft trajectories, that the inclination of the mean trajectory changes only slowly with the time, it follows from equation (3.215) of Fowler <u>et al.</u> that y and z are the real and complex parts, respectively, of the time integral of $\varkappa\eta$, namely

$$\frac{\varkappa K_1}{in_1 - \lambda_1} e^{in_1 t - \lambda_1 t} + \frac{\varkappa K_2}{in_2 - \lambda_2} e^{in_2 t - \lambda_2 t}.$$

The λ's are very small compared with the n's, and thus <u>y</u> and <u>z</u> perform vibrations similar to <u>j</u> and <u>k</u>, with amplitudes equal to the amplitudes of <u>j</u> and <u>k</u> multiplied by $\varkappa/n_1$ for the nutation and $\varkappa/n_2$ for the precession, and with phases that lag 90° behind.

Consider a bullet that starts its motion with an angle of yaw δ_0, an orientation angle $\varphi = 90°$, and with $\dot{\delta}$ and $\dot{\varphi}$ both initially equal to zero. This situation arises when shooting to starboard horizontally from a moving aircraft. The diagram below shows



that

$$a_2 - a_1 = \delta_o$$

$$n_2 a_2 = n_1 a_1$$

and thus

$$a_1 = n_2\delta_o/(n_1 - n_2); \quad a_2 = n_1\delta_o/(n_1 - n_2).$$

The orientation angle of the precession is 90°, of the nutation 270°, initially. Thus the initial value of z is zero, and the initial value, Δ, of y is

$$\Delta = \varkappa\delta_o\left[\frac{n_1}{n_2}\,\frac{1}{n_1-n_2} - \frac{n_2}{n_1}\,\frac{1}{n_1-n_2}\right]$$

$$= \varkappa\delta_o\,\frac{n_1+n_2}{n_1 n_2}$$

$$= \varkappa\delta_o\,\frac{4bS}{[illegible]}$$

$$= \frac{A}{md}\,\frac{K_L}{K_M}\,\frac{\delta_o}{u_o} \qquad (12)$$

by equation (10). Since the tangent to the instantaneous trajectory points higher than the tangent to the mean trajectory by this amount at the instant of firing, it follows that the tangent to the mean trajectory is lower than the bore of the gun by the angle Δ given by (12). This is the so-called "windage jump". Its amount, for a given bullet, depends essentially only on the initial angle of yaw and on the initial air speed of the bullet, u_o.

The preceding derivation of the formula (12) for windage jump starts from equation (3.215) of Fowler et al. The following alternative derivation of the same equation (12), although it is less elegant and rigorous, and is somewhat longer, has the advantage of being more intuitively obvious. The components of the

cross wind force in the directions $\underline{O2}$ and $\underline{O3}$ give rise to accelerations of the center of mass

$$\dot{y} = \varkappa(-a_2 \sin n_2 t + a_1 \sin n_1 t)$$

$$\dot{z} = \varkappa(a_2 \cos n_2 t - a_1 \cos n_1 t)$$

if damping is neglected. Integrating,

$$y = \varkappa\left(\frac{a_2}{n_2} \cos n_2 t - \frac{a_1}{n_1} \cos n_1 t\right)$$

$$z = \varkappa\left(\frac{a_2}{n_2} \sin n_2 t - \frac{a_1}{n_1} \sin n_1 t\right)$$

where the constants of integration are zero because the quantities $\underline{y}$ and $\underline{z}$ can contain no constant parts, being pure vibrations that may or may not be damped. More accurately, the mean trajectory with respect to which $\underline{y}$ and $\underline{z}$ are measured is so defined that $\underline{y}$ and $\underline{z}$ contain purely vibrational terms. The last pair of equations show that initially $\underline{z}$ is zero, and that the initial value of $\underline{y}$ is

$$\Delta = \varkappa\left(\frac{a_2}{n_2} - \frac{a_1}{n_1}\right)$$

$$= \varkappa\delta_o\left(\frac{n_1 + n_2}{n_1 n_2}\right)$$

$$= \frac{AN}{md}\frac{K_L}{K_M}\frac{\delta_o}{u_o}$$

as before. The inclusion of damping factors $e^{-\lambda t}$ can be shown not to modify the last result if the λ's are small in comparison to the $\underline{n}$'s.

5. The Yawing Motion of a Bullet Fired from an Aircraft. Denote the initial angle of yaw by δ_o. The time derivatives of both δ and φ are initially zero, and therefore it follows from the diagram in Section 3 that

$$a_2 - a_1 = \delta_o,$$

$$n_2 a_2 - n_1 a_1 = 0$$

whence

$$a_1 = n_2\delta_o/(n_1 - n_2)$$

$$a_2 = n_1\delta_o/(n_1 - n_2)$$

initially, and thus from equations (10)

$$a_1 = (1 - p_o)\delta_o/2p_o$$

$$a_2 = (1 + p_o)\delta_o/2p_o$$

initially, where p_o is the initial value of $\underline{p}$. Now according to the discussion in Section 3, if the bullet is of the type (like all the small arms bullets studied, and the 37mm M54 projectile) with a single damping factor for both precession and nutation, it follows that a_1 and a_2 are given at all times by

$$a_1 = \sigma(1 - p_o)\delta_o/2p_o$$

$$a_2 = \sigma(1 + p_o)\delta_o/2p_o$$

where

$$\sigma = (p_o/p)^{\frac{1}{2}} e^{-\rho_a aP} \quad . \tag{13}$$

The value of δ^2 is given by (11), from which it follows that the mean value of δ^2, averaged over a single period of δ^2, is

$$\begin{aligned} \overline{\delta^2} &= a_1^2 + a_2^2 \\ &= \delta_o^2\sigma^2 \frac{1 + p_o^2}{2p_o^2} \\ &= \delta_o^2\sigma^2 \frac{s_o - \frac{1}{2}}{s_o - 1} \end{aligned} \tag{14}$$

where s_o is the initial value of the stability factor $\underline{s}$.

The stability factor $\underline{s}$ varies along the trajectory very closely as $1/u^2$ where $\underline{u}$ is the remaining velocity, and thus

$$p^2 = 1 - u^2/u_o^2 s_o$$

and expanding in powers of $\underline{P}$

$$\begin{aligned} p^2 &= p_o^2 - (2/u_o s_o)(du/dP)_o P + \ldots\ldots\ldots \\ &= p_o^2\left(1 + \frac{2\rho_a d^2}{(s_o - 1)m} K_D P\right) + \ldots\ldots\ldots \end{aligned}$$

so

$$\frac{p_o}{p} = e^{-\frac{\rho_a d^2}{s_o-1}\,\frac{1}{m}\,K_D P}$$

approximately, where ρ_a is the air density, $\underline{m}$ is the mass, $\underline{d}$ is the diameter, and K_D' is the drag coefficient. Hence

$$\sigma^2 = e^{-2\rho_a P(a + a')}$$

where $\underline{a}$ is given by (11A) and

$$a' = \frac{d^2}{2(s_o - 1)m} K_D. \tag{15}$$

Accordingly,

$$\overline{\delta^2} = \delta_o^2 \frac{s_o - \frac{1}{2}}{s_o - 1} e^{-2\rho_a P(a + a')} . \tag{16}$$

6. The Yaw-drag Effect. The yaw of the bullet has two effects upon the trajectory: It renders the drag greater than it would be in the absence of yaw, and it introduces the "windage jump" evaluated in section 4. In the present section we consider the effect upon the trajectory of the increased drag occasioned by the motion about the center of mass. The effect of the "windage jump" will be considered in the next section.

Let $\underline{q}$, $\underline{r}$ be rectangular axes fixed in the air, with $\underline{q}$ vertical and $\underline{r}$ horizontal, the plane $\underline{q}$, $\underline{r}$ containing the line of departure (windage jump being here neglected) in the air, and the origin being the position of the gun at the instant of firing. The equations of motion of the yawing bullet, as influenced by gravity and drag, are

$$\begin{aligned} \ddot{q} + \rho \frac{G_n}{C_n}(1 + \overline{\delta^2} K_{D\delta})\dot{q} + g &= 0 \\ \ddot{r} + \rho \frac{G_n}{C_n}(1 + \overline{\delta^2} K_{D\delta})\dot{r} &= 0 \end{aligned} \tag{17}$$

if periodic terms are neglected whose mean values, over their periods, are zero and whose effects remain small at all ranges. Here ρ is the relative air density, G_n is the appropriate drag function for zero yaw, C_n is the appropriate ballistic coefficient, $K_{D\delta}$ is the yaw-drag coefficient, $\underline{g}$ is the acceleration of gravity,

δ^2 is given by equation (16) and G_n may be evaluated, in accordance with the Siacci approximation, for the argument $u = \dot{P}$. As in section 2, ρ may be given its muzzle value. Then one has

$$P = r \sec \theta_o$$

$$Q = r \tan \theta_o - q$$

where θ_o is the initial inclination, and so it follows from equations (17) that

$$\frac{du}{G_n} + \rho \frac{dP}{C_n} (1 + \overline{\delta^2} K_{D\delta}) = 0 \tag{18}$$

$$\ddot{Q} - \frac{d\dot{P}}{dP} \dot{Q} = g. \tag{19}$$

The first equation, (18), may be integrated at once, use being made of (16). One obtains the equation

$$S = S_o + \frac{\rho}{C_n} P + \frac{1}{2C_n c} K_{D\delta}\, \delta_o^2 \frac{s_o - \frac{1}{2}}{s_o - 1} (1 - e^{-2\rho c P}) \;\ldots. \tag{20}$$

where

$$c = (a + a')\rho_s.$$

The symbol ρ_s denotes the standard density, 0.07513 pounds per cubic foot, and arises because the symbol ρ_a in this Report denotes air density; while the symbol ρ denotes the relative air density, in units of ρ_s. In equation (20), S is the Siacci "S" function of <u>u</u>. The quantity <u>a</u> is given by equation (11A), and <u>a</u>' by (15). Let

$$c' = a\rho_s$$

$$= (f_s + \varkappa_s)/2u_s$$

where f_s and $\varkappa_s$ are the values of f and $\varkappa$ at standard density when the velocity is u_s. Also, let

$$c'' = \rho_s d^2 K_D/2m$$

$$= G_n(v_o)/2C_n v_o;$$

and then

$$c = c' + c''/(s_o - 1). \tag{20A}$$

The best value of K_{D_0} at present available appears to be approximately 16.4.

The initial yaw δ_0, to the accuracy with which the theory of the yawing motion has been worked out, is equal to its sine, so that

$$\delta_0^2 = w^2(1 - \sin^2 Z \cos^2 A)/u_0^2 \qquad (21)$$

while

$$s_0 = v_0^2 s_s/\rho u_0^2 \qquad (22)$$

where s_s is the value of the stability factor $\underline{s}$ near the muzzle of a stationary gun in air of standard density. For any P, equation (20) allows S, and hence $\underline{u}$, to be determined. The equation

$$t = \int_0^P (1/u)\,dP \qquad (23)$$

allows $\underline{t}$ to be determined by a quadrature.

The auxiliary equation of equation (19) is linear in Q, and (19) may be integrated once to obtain

$$\dot{Q} = gu\int_0^P \frac{dP}{u^2}$$

and again to obtain

$$Q = g\int_0^P \int_0^P \frac{dP}{u^2}\,dP. \qquad (24)$$

Once t and Q are determined, equations (2), (3), and (4) may be employed to determine the vertical and lateral deflections, μ and λ, just as for a trajectory undisturbed by yaw, and also the range $\underline{D}$.

The preceding results, equations (20) and (24) in particular, are of a general type first obtained by R. H. Kent and L. S. Dederick. Kent showed how the yaw-drag term could be dealt with

by the Siacci method so as to yield $\underline{u}$ in terms of P, through an equation of the type (20). Dederick showed how Q could be obtained in the form of a double quadrature, (24). The particular results presented here, however, are new. Formerly it was thought that the precessional and nutational damping rates for small arms bullets, including 37mm. were different; the nutational being supposed to be more rapid. Thus δ^2, in equation (18), was given an incorrect value instead of the value given by (16). The constant $\underline{c}$, in (20), was also given an incorrect value. These errors, first revealed through aircraft firings conducted by Colonel Leslie E. Simon and the author (their experimental results have not yet been published), date back to the paper of Fowler et al. who assumed incorrectly that the value of γ was so small as to be negligible. Here the author has applied the elegant computational procedures of Kent and Dederick to a more correctly specified yawing motion.

7. <u>The Effect of Windage Jump Upon the Trajectory</u>. The windage jump Δ is small, and its effects may be applied to the trajectory by simple, yet closely approximate, methods. Consider the set of axes x', y', z' of section 2, at rest in the air. The windage jump changes the direction of the line of departure by adding to u_o, regarded as a vector, a vector equal numerically to $u_o\Delta$ and directed at right angles to the plane containing the z' axis and the bore of the gun, in a sense given by a right-handed rotation about the z' axis. The additional vector has direction cosines in the primed system, therefore, of $-\sin Z \sin A/(1-\sin^2 Z\cos^2 A)^{\frac{1}{2}}$, $\cos Z/(1 - \sin^2 Z \cos^2 A)^{\frac{1}{2}}$, 0 and components equal to the preceding direction cosines multiplied by $u_o\Delta$. The windage jump does not alter the magnitude of u_o and thus the effect of the jump is to increase x, y, z, of the bullet by the quantities:

$$-P\Delta\sin Z\sin A/(1 - \sin^2 Z\cos^2 A)^{\frac{1}{2}},\ P\Delta\cos Z/(1 - \sin^2 Z\cos^2 A)^{\frac{1}{2}},\ 0.$$

It follows that ξ is not altered, that η is increased by

$$-P\ \Delta\ \sin A/(1 - \sin^2 Z\cos^2 A)^{\frac{1}{2}}$$

and that ζ is increased by

$$P\ \Delta\ \cos Z\cos A/(1 - \sin^2 Z\cos^2 A)^{\frac{1}{2}}.$$

Since one is concerned with small angular deflections, the effect of the windage jump is thus to increase λ and μ by the amounts, in radians

$$\delta\lambda = \frac{P}{D}\ \Delta\ \cos Z\cos A/(1 - \sin^2 Z\cos^2 A)^{\frac{1}{2}}$$

$$\delta\mu = -\frac{P}{D}\ \Delta\ \sin A/(1 - \sin^2 Z\cos^2 A)^{\frac{1}{2}}$$

while the range, D, is not altered (to any significant extent).

If one makes use of equation (12) for Δ, and makes use of equation (21) for u_o, one thus finds that the effects in radians of windage jump on the horizontal and vertical deflections are

$$\delta\lambda = b\,\frac{\rho}{D}\cos Z\cos A\,\frac{w}{u_o^2}$$

$$\delta\mu = -b\,\frac{\rho}{D}\sin A\,\frac{w}{u_o^2} \qquad \ldots (25)$$

where

$$b = \frac{AN}{md}\,\frac{K_L}{K_M}\,. \qquad \ldots (26)$$

8. Computing Procedures. The sequence of operations is to find u_o by equation (1); δ_o^2 by (21), in (radians)^2; s_o by (22); c by (20A); and then find the Siacci "S" by (20). From S., u is found; then t and Q are computed by quadratures in accordance with (23) and (24). Then ξ, η, ζ are found by equations (2); D by (3) if ξ is not sufficiently accurate as an approximation to D; and λ and μ by equations (4). With close accuracy under most conditions, λ and μ are given directly in mils* as the right-hand members of equations (4) multiplied by 1020. Finally, the effects of windage jump, $\delta\lambda$ and $\delta\mu$, are computed by equations (25) and added to λ and μ already computed without jump, in order to obtain the values with jump. Equations (25) will yield values in mils directly if for b a value is used equal to the value given by (26) multiplied by 1020.

In the preceding equations, ρ always stands for the relative air density.

The value of the constants s_s, c', c", b, $K_{D\delta}$ are given below, for two gun-projectile combinations. (1) is the Cal. 0.50 AP M2 projectile fired at 2700 feet per second from rifling having a twist of one turn in 15 inches. (2) is the 37mm H.E. shell M54 or practice projectile M55 with P.D. fuze M56, fired at 2000 feet per second from rifling having a twist of one turn in 25 calibers. The fundamental data have been furnished by H. P. Hitchcock.

Combination	s_s	c' ft^{-1}	c" ft^{-1}	b mils ft/sec	$K_{D\delta}$
(1)	2.17	.00208	.000076	45,800	16.4
(2)	2.39	.00172	.000066	38,000	16.4

*One mil is 1/6400 revolutions.

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